Nonlinear Analysis of Pretwisted Rods Using "Principal Curvature Transformation" Part I: Theoretical Derivation

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A numerical method for analyzing the nonlinear coupled bending-torsion of pretwisted rods is derived. This method combines use of principal curvature transformation and the technique of generalized coordinates, and a very efficient tool results. The two main advantages are the ease with which different kinds of nonlinear models incorporating different magnitudes of nonlinearities can be derived and implemented and reduced computing time required as compared with other methods for parametric studies where the loads, structural properties, and pretwist are varied. The present derivation includes three versions of a model for small strains and moderate elastic rotations. A fourth model, where products of the elastic rotations are not negligible compared with unity, is also presented. The second part of the paper will include numerical examples using the model described herein.

I. Introduction

RODS are very important structural elements in all types of engineering systems. In many applications, these rod elements undergo finite deformations that require a nonlinear approach to their structural analysis. Over the years, different methods of analyzing the nonlinear behavior of rods have been developed. It is beyond the scope of this paper to present a review of previous work. The interested reader may find more details in Refs. 1-3 and other sources.

Recently, the authors of the present paper have presented a computationally efficient method for analyzing the linear bending of pretwisted blades.[4] The approach combined use of a specially defined system of coordinates and a generalized coordinates technique. A system stiffness matrix is thereby obtained without direct structural modeling of the pretwisted blade. This is achieved by using the mode shapes of the same blade, but untwisted, as generalized coordinates. These modes and frequencies could be obtained from numerical or analytic calculations or from experiments.

The purpose of the present paper is to show how the basic approach of Ref. 4 can be extended to include the case of nonlinear coupled bending-torsion of pretwisted rods. Practical structural elements for which such analysis is important are helicopter blades, very high aspect ratio wings, or very limber space structures. The first part of the paper presents the derivation of the model. The second part will present numerical results and comparisons with experiments.

II. Basic Geometric Relations

Consider a slender rod having a straight elastic axis. A Cartesian system of coordinates x-y-z is defined, where x coincides with the elastic axis of the rod, and y and z are cross-sectional coordinates. The unit vectors $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ are in the directions of the coordinate lines x, y, and z, respectively. Each material point of the rod is defined by its coordinates in the undeformed state—x, y, z.

As a result of loads acting along the rod, considered to be a slender body, it deforms. The Bernoulli-Euler hypothesis is assumed to apply; that is, plane cross sections perpendicular to the elastic axis before deformation remain plane and perpendicular to the deformed elastic axis after deformation, and strains and stresses within these cross sections are neglected. (More regarding this hypothesis may be found in Ref. 5.) Thus, as a result of the deformation, the triad $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$ is transformed into a new triad $\hat{e}_{x1}, \hat{e}_{y1}, \hat{e}_{z1}$. It is then possible to express the position vector to any point in the deformed rod as follows:

$$\bar{R} = (x+u)\,\hat{e}_x + v\hat{e}_y + w\hat{e}_z + y\hat{e}_{y1} + z\hat{e}_{z1} + \tilde{\psi}\hat{e}_{x1} \qquad (1)$$

Here u, v, and w are the displacement components (in the x, y, and z directions, respectively) of the point x on the elastic axis. They are accompanied by a rotation ϕ of the cross section about the elastic axis. The term $\tilde{\psi}\hat{e}_{x1}$ represents warping of the cross sections and is the standard extension of the Bernoulli-Euler hypothesis, which makes it possible to consider coupled bending-torsion phenomena.

If Euler angles are used to describe the elastic rotations, then it is important that the sequence of the elastic deformation be consistent with the definitions, if moderate elastic rotations are to be accounted for (for more details, see Refs. 1 and 5–7). The present derivation will assume deformations in the following sequence: u, v, w, ϕ (the same as in Refs. 1, 3, 6, and 7. For more details see Appendix C of Ref. 17.). Note that the sequence is only tied to the mathematical modeling, therefore, if a particular final state of deformation is described by some different sequence, the same distribution of displacements and moments will be obtained within the accuracy of the model. The present derivation will be restricted to the usual cases where warping of cross sections is small enough so that its contributions can be ignored except for their influence on the torsional rigidity. Influences of warping due to pretwist[8,9] are also neglected. These effects may be important in special cases of highly pretwisted rods having thin open cross sections, but are not dealt with here. Whenever necessary, the torsional rigidity can be adjusted to take account of such effects. As an extension of the classical Saint Venant torsion

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theory, it is assumed that

$$\tilde{\psi} = T\psi \tag{2}$$

where T is the twist (in this case, of the deformed elastic axis) and ψ is the Saint Venant warping function of the particular cross section. Use of this assumption in Refs. 1 and 9 achieved very good agreement between the theoretical and experimental results.

By using Eqs. (1) and (2) and the definition of strain, the following expressions for the strain components are obtained (for more details, see Eqs. (3.3–3.8) of Ref. 7 and specialize for the present case in which the initial curvature is zero):

$$\epsilon_{xx} = \tilde{\epsilon}_{xx} - yK_y - zK_z = \tilde{\epsilon}_c - (y - y_c)K_y - (z - z_c)K_z \tag{3a}$$

$$\epsilon_{xy} = \tfrac{1}{2}(\psi_{,y} - z)T \tag{3b}$$

$$\epsilon_{xz} = \tfrac{1}{2}(\psi_{,z} + y)T \tag{3c}$$

Here only shear strains due to torsion are accounted for, neglecting transverse shear deformations, consistent with a long slender rod assumption, and all strains are assumed small and negligible compared to unity. The normal strain of the deformed elastic axis, which appears in Eq. (3a), is given by

$$\tilde{\epsilon}_{xx} = u_{,x} + \tfrac{1}{2}(v_{,x}^2 + w_{,x}^2) \tag{4}$$

The term ϵ_c, which also appears in Eq. (3a), is the normal strain at the cross-sectional tension center, which is defined by the coordinates (y_c, z_c) and is also called, in may cases, the neutral point of the cross section. Here, K_y and K_z are the curvature components. They are defined by the following relations:

$$\hat{e}_{x1,x} = K_y\hat{e}_{y1} + K_z\hat{e}_{z1} \tag{5a}$$

$$\hat{e}_{y1,x} = -K_y\hat{e}_{x1} + T\hat{e}_{z1} \tag{5b}$$

$$\hat{e}_{z1,x} = -K_z\hat{e}_{x1} - T\hat{e}_{y1} \tag{5c}$$

III. Strain Energy of the Deformed Rod

If it is assumed that the rod is made of a linear elastic material, then the general expression for strain energy is

$$U_E = \frac{1}{2}\int_0^L \iint_A (\sigma_{xx}\epsilon_{xx} + 2\tau_{xy}\epsilon_{xy} + 2\tau_{xz}\epsilon_{xz})\,\mathrm{d}A\,\mathrm{d}x \tag{6}$$

where A is the rod cross-sectional area and L the rod's length. In Eq. (6), use has been made of the Bernoulli-Euler hypothesis, according to which stresses and strains lying within the planes of the rod's cross sections are neglected. For the case of slender rods, the following constitutive relations are used:

$$\sigma_{xx} = E\epsilon_{xx} \tag{7a}$$

$$\tau_{xy} = 2G\epsilon_{xy} \tag{7b}$$

$$\tau_{xz} = 2G\epsilon_{xz} \tag{7c}$$

where σ_{xx}, τ_{xy}, and τ_{xz} are the normal and shearing components of the stress tensor, and E and G are the local Young's modulus and shear modulus, respectively.

Now Eqs. (7) are substituted into Eq. (6), and the strain components are expressed by Eqs. (3). Integration of the expression for U_E over the cross section of the rod yields the following equation:

$$U_E = \frac{1}{2}\int_0^L \big[(EA)\epsilon_c^2 + (EI_{yy})K_y^2 + 2(EI_{yz})K_yK_z + (EI_{zz})K_z^2 + (GJ)T^2\big]\,\mathrm{d}x \tag{8}$$

where

$$(EI_{yy}) = \iint_A E(y - y_c)^2\,\mathrm{d}A \tag{9a}$$

$$(EI_{zz}) = \iint_A E(z - z_c)^2\,\mathrm{d}A \tag{9b}$$

$$(EI_{yz}) = \iint_A E(y - y_c)(z - z_c)\,\mathrm{d}A \tag{9c}$$

$$(GJ) = \iint_A G\big[y^2 + z^2 + y\psi_{,z} - z\psi_{,y}\big]\,\mathrm{d}A = \iint_A G\big[(\psi_{,y} - z)^2 + (\psi_{,z} + y)^2\big]\,\mathrm{d}A \tag{9d}$$

$$y_c = \frac{1}{A}\iint_A y\,\mathrm{d}A \tag{9e}$$

$$z_c = \frac{1}{A}\iint_A z\,\mathrm{d}A \tag{9f}$$

The quantities (EI_{yy}), (EI_{zz}), and (EI_{yz}) are components of the cross-sectional bending stiffness calculated about the center of tension. Equations (9e) and (9f) are, in fact, definitions of the tension center. The quantity (GJ) is the torsional stiffness calculated about the elastic axis.

Since the present analysis deals with the coupled bending-torsion of a slender rod as uncoupled from extensile effects, axial deformations may be neglected by assuming inextensionality of the rod. In this case, ϵ_c is taken equal to zero and the first term inside the integral in Eq. (8) disappears. If necessary, this term can be retained without causing any major difficulties in the derivations.

As shown in Fig. 1, it is possible to define, at each cross section, a pair of principal directions η and ζ, which are rotated relative to the axes y and z, respectively (i.e., about the x axis), by an angle θ. The local bending stiffness components relative to these principal directions are

$$(EI_{\eta\eta}) = \iint_A E\eta^2\,\mathrm{d}A \tag{10a}$$

$$(EI_{\zeta\zeta}) = \iint_A E\zeta^2\,\mathrm{d}A \tag{10b}$$

and $(EI_{\eta\zeta})$ is zero, by the definition of principal directions.

If (EI_{yy}), (EI_{zz}), and (EI_{yz}) are expressed as functions of $(EI_{\eta\eta})$, $(EI_{\zeta\zeta})$, and θ, and these expressions are substituted into Eq. (8) (for the detailed derivation, see Ref. 4), the

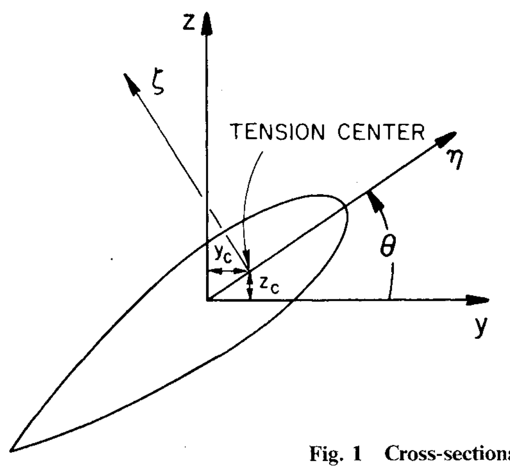

Fig. 1 Cross-sectional coordinates.

following expression for U_E is obtained:

$$U_E = \frac{1}{2}\int_0^L \Big[(EI_{\eta\eta})(K_y \cos\theta + K_z \sin\theta)^2 + (EI_{\zeta\zeta})(-K_y \sin\theta + K_z \cos\theta)^2 + (GJ)T^2 \Big] \, dx \tag{11}$$

The "principal curvatures" are defined here as the square roots of the terms that multiply $(EI_{\eta\eta})$, $(EI_{\zeta\zeta})$, and (GJ) in Eq. (11). For reasons that will become clear later in the derivation, the "principal curvatures" are expressed as the derivatives of the "principal displacements," v_e, w_e, and ϕ_e. It should be emphasized that these displacement components do not have a simple geometric meaning like the displacement components u, v, w, and ϕ, which were defined previously. The "principal displacements" emerge from the definition of the "principal curvatures." That is, v_e, w_e, and ϕ_e are defined by (the appropriate integrals of) the following equations:

$$v_{e,xx} = K_y \cos\theta + K_z \sin\theta \tag{12a}$$

$$w_{e,xx} = -K_y \sin\theta + K_z \cos\theta \tag{12b}$$

$$\phi_{e,x} = T \tag{12c}$$

Using these definitions, the expression for strain energy becomes

$$U_E = \frac{1}{2}\int_0^L \Big[(EI_{\eta\eta}) v_{e,xx}^2 + (EI_{\zeta\zeta}) w_{e,xx}^2 + (GJ)\phi_{e,x}^2 \Big] \, dx \tag{13}$$

Although, as mentioned previously, the principal displacement components do not have a clear geometric meaning; Eq. (13) is clearly analogous to the strain energy of a straight rod having zero pretwist. In this instance, the system of coordinates x, η, and ζ is a Cartesian system of principal coordinates for all cross sections; v_e and w_e are the components in the directions η and ζ, respectively, of small transverse displacement; and ϕ_e is a small elastic rotation. This case will be denoted the "analog rod" in what follows. The spanwise distribution of $(EI_{\eta\eta})$, $(EI_{\zeta\zeta})$, and (GJ) of the "analog rod" is identical to the spanwise distributions of the same variables of the real rod. The advantage of Eq. (13) is that there is no elastic coupling between the two bending components, nor between bending and torsion.

At this stage, the method of generalized coordinates is applied. According to this method, the displacement components v_e, w_e, and ϕ_e are described by the following series:

$$v_e = \sum_{j=1}^{N_{ve}} q_{ve(j)} FV_{e(j)} \tag{14a}$$

$$w_e = \sum_{k=1}^{N_{we}} q_{we(k)} FW_{e(k)} \tag{14b}$$

$$\phi_e = \sum_{\ell=1}^{N_{\phi e}} q_{\phi e(\ell)} F\phi_{e(\ell)} \tag{14c}$$

Here, $FV_{e(j)}$, $FW_{e(k)}$, and $F\phi_{e(\ell)}$ are shape functions that form the generalized coordinates. Although it may not always be a necessary condition, it seems that it will be beneficial to choose the shape functions such that $FV''_{e(j)}$, $FW''_{e(k)}$, and $F\phi'_{e(\ell)}$ (upper prime denotes differentiation with respect to x) satisfy the natural boundary conditions in cases when such conditions exist at one or the other of the two boundaries.

Substitution of Eqs. (14) into Eq. (13) results in:

$$U_E = \frac{1}{2}\Bigg[\sum_{j=1}^{N_{ve}} \sum_{j1=1}^{N_{ve}} B1_{(j,j1)} q_{ve(j)} q_{ve(j1)} + \sum_{k=1}^{N_{we}} \sum_{k1=1}^{N_{we}} B2_{(k,k1)} q_{we(k)} q_{we(k1)} + \sum_{\ell=1}^{N_{\phi e}} \sum_{\ell 1=1}^{N_{\phi e}} B3_{(\ell,\ell 1)} q_{\phi e(\ell)} q_{\phi e(\ell 1)} \Bigg] \tag{15}$$

where

$$B1_{(j,j1)} = \int_0^L (EI_{\eta\eta}) FV''_{e(j)} FV''_{e(j1)} \, dx \tag{16a}$$

$$B2_{(k,k1)} = \int_0^L (EI_{\zeta\zeta}) FW''_{e(k)} FW''_{e(k1)} \, dx \tag{16b}$$

$$B3_{(\ell,\ell 1)} = \int_0^L (GJ) F\phi'_{e(\ell)} F\phi'_{e(\ell 1)} \, dx \tag{16c}$$

There is a special set of shape functions that present certain advantages,[4] namely, where $FV_{e(j)}$ is the jth natural mode of vibration of the "analog rod" in the η direction, and $\omega_{ve(j)}$ is the natural frequency of that mode; $FW_{e(k)}$ is the kth natural mode of vibration of the "analog rod" in the ζ direction, and $\omega_{we(k)}$ is the frequency of this mode; and $F\phi_{e(\ell)}$ is the ℓth mode of torsional vibrations of the "analog rod," and $\omega_{\phi(\ell)}$ is its frequency. All of the above mentioned modes and frequencies are functions of the stiffness and mass distributions. The stiffness distributions of the analog rod have been defined previously in Eqs. (9d) and (10). The mass and mass moments of inertia per unit length of the "analog rod" are chosen equal to the same variables of the "real rod." While in the case of the real rod there may exist an offset between the elastic axis and the center-of-mass location, in the analog rod they are assumed to coincide. This provides another advantage; i.e., there can be no coupling between the transverse (bending) vibrations in the η and ζ directions or between the transverse and torsional vibrations of the analog rod so defined. It should be emphasized that the use of the analog rod applies only to the strain energy (structural contributions). Later, when the present method is applied to the dynamic behavior of rods,[16] an offset between the elastic axis and center of mass is included among other effects that influence the dynamic behavior. Its effect then appears in kinetic-energy expressions.

If the orthogonality properties of the modes of free vibrations[10,11] are taken into account, then after integration by parts of Eqs. (16) the following simplified expression for the strain energy is obtained:

$$U_E = \frac{1}{2}\Bigg[\sum_{j=1}^{N_{ve}} q_{ve(j)}^2 \omega_{ve(j)}^2 M_{ve(j)} + \sum_{k=1}^{N_{we}} q_{we(k)}^2 \omega_{we(k)}^2 M_{we(k)} + \sum_{\ell=1}^{N_{\phi e}} q_{\phi e(\ell)}^2 \omega_{\phi e(\ell)}^2 M_{\phi e(\ell)} \Bigg] \tag{17}$$

in which the generalized mass expressions are defined as

$$M_{ve(j)} = \int_0^L FV_{e(j)}^2 m \, dx \tag{18a}$$

$$M_{we(k)} = \int_0^L FW_{e(k)}^2 m \, dx \tag{18b}$$

$$M_{\phi e(\ell)} = \int_0^L F\phi_{e(\ell)}^2 MI_p \, dx \tag{18c}$$

where m and MI_p are the mass and mass polar moment of inertia about the elastic axis per unit length.

If a vector of coefficients $\{q_e\}$ is defined as follows,

$$\{q_e\}^t = \langle q_{ve(1)}, \ldots, q_{ve(j)}, \ldots, q_{ve(N_{ve})}, q_{we(1)}, \ldots, q_{we(k)}, \ldots, q_{we(N_{we})}, q_{\phi e(1)}, \ldots, q_{\phi e(\ell)}, \ldots, q_{\phi e(N_{\phi e})} \rangle \quad (19)$$

then the strain energy can be expressed using the following matrix notation:

$$U_E = \tfrac{1}{2}\{q_e\}^t [K_e]\{q_e\} \quad (20)$$

The generalized stiffness matrix $[K_e]$ is of order N_e, where

$$N_e = N_{ve} + N_{we} + N_{\phi e} \quad (21)$$

This matrix is assembled of three submatrices, as follows:

$$[K_e] = \begin{bmatrix} [B1] & 0 & 0 \\ 0 & [B2] & 0 \\ 0 & 0 & [B3] \end{bmatrix} \quad (22)$$

The terms of submatrices $[B1]$, $[B2]$, and $[B3]$ are defined by Eqs. (16a–c), respectively.

In the case where the natural mode shapes of free vibrations of the analog rod are chosen as the generalized coordinates, the matrix $[K_e]$ becomes a diagonal matrix, where the terms along the diagonal are simply the generalized masses times the square of the corresponding natural frequencies of the analog rod; i.e.,

$$K_{e(i,i)} = \omega^2_{ve(i)} M_{ve(i)}, \qquad \text{for } 1 \le i \le N_{ve}$$

$$= \omega^2_{we(i-N_{ve})} M_{we(i-N_{ve})}, \qquad \text{for } N_{ve} < i \le (N_{ve} + N_{we})$$

$$= \omega^2_{\phi e(i-N_{ve}-N_{we})} M_{\phi e(i-N_{ve}-N_{we})}, \qquad \text{for } (N_{ve}+N_{we}) < i \le N_e \quad (23)$$

IV. Transformation Between "Principal Displacement" Components and Cartesian Components

The curvature and twist of the rod are functions of its displacements and their derivatives with respect to x. As the magnitude of the deformation increases, expressions involving these quantities, which will yield acceptable predictions, become more complicated. In the case of small strains and moderate elastic rotations (strains and products of elastic rotations are neglected compared to unity), the relations are [see Eqs. (2.28–2.30) of Ref. 7, as applied to a straight, i.e., zero curvature, rod]:

$$K_y = v_{,xx} + w_{,xx}\phi \quad (24a)$$

$$K_z = w_{,xx} - v_{,xx}\phi - \underline{v_{,xx}v_{,x}w_{,x}} \quad (24b)$$

$$T = \phi_{,x} + v_{,xx}w_{,x} \underline{- v_{,xx}v_{,x}\phi + w_{,xx}w_{,x}\phi - v_{,x}w_{,xx}\phi^2} \quad (24c)$$

In many practical cases the underlined terms in Eqs. (24) are small and can be neglected, as has been done in Refs. 1 and 5. In the present derivation, they will be retained and their influence investigated later. The possibility of retaining these terms without complicating the analysis excessively is one of the advantages of the present method compared to others. Substitution of Eqs. (24) into Eqs. (12) yields

$$v_{e,xx} = (v_{,xx} + w_{,xx}\phi)\cos\theta + (w_{,xx} - v_{,xx}\phi - v_{,xx}v_{,x}w_{,x})\sin\theta \quad (25a)$$

$$w_{e,xx} = -(v_{,xx} + w_{,xx}\phi)\sin\theta + (w_{,xx} - v_{,xx}\phi - v_{,xx}v_{,x}w_{,x})\cos\theta \quad (25b)$$

$$\phi_{e,x} = \phi_{,x} + v_{,xx}w_{,x} - v_{,xx}v_{,x}\phi + w_{,xx}w_{,x}\phi - v_{,x}w_{,xx}\phi^2 \quad (25c)$$

It should be mentioned that a similar transformation approach has been used by Bielawa[12] although the interpretation there is different from the present one.

The last system of nonlinear ordinary differential equations does not have a general analytic solution. Analytic solutions are probably available only for special linear cases.[4] An approximate numerical solution of Eqs. (25) therefore appears necessary. These equations can be solved, for example, using a finite difference approach, but since v_e, w_e, and ϕ_e have already been expressed as a series of functions, it seems only natural to express v, w, and ϕ in the same way, i.e.,

$$v = \sum_{j=1}^{N_v} q_{v(j)} FV_{(j)} \quad (26a)$$

$$w = \sum_{k=1}^{N_w} q_{w(k)} FW_{(k)} \quad (26b)$$

$$\phi = \sum_{\ell=1}^{N_\phi} q_{\phi(\ell)} F\phi_{(\ell)} \quad (26c)$$

Note that the shape functions $FV_{e(j)}$, $FW_{e(k)}$, and $F\phi_{e(\ell)}$ in Eqs. (14) and those in Eqs. (26), i.e., $FV_{(j)}$, $FW_{(k)}$, and $F\phi_{(\ell)}$ are shown as being different. In practice, the same functions will usually suffice.

Substitution of Eqs. (14) and (26) into Eqs. (25) results in the following system:

$$\sum_{j=1}^{N_{ve}} q_{ve(j)} FV''_{e(j)} = \sum_{j=1}^{N_v} q_{v(j)} FV''_{(j)} \cos\theta + \sum_{k=1}^{N_w} q_{w(k)} FW''_{(k)} \sin\theta + \sum_{k=1}^{N_w}\sum_{\ell=1}^{N_\phi} q_{w(k)} q_{\phi(\ell)} FW''_{(k)} F\phi_{(\ell)} \cos\theta - \sum_{j=1}^{N_v}\sum_{\ell=1}^{N_\phi} q_{v(j)} q_{\phi(\ell)} FV''_{(j)} F\phi_{(\ell)} \sin\theta - \sum_{j=1}^{N_v}\sum_{j1=1}^{N_v}\sum_{k=1}^{N_w} q_{v(j)} q_{v(j1)} q_{w(k)} FV''_{(j)} FV'_{(j1)} FW'_{(k)} \cos\theta \quad (27a)$$

$$\sum_{k=1}^{N_{we}} q_{we(k)} FW''_{e(k)} = -\sum_{j=1}^{N_v} q_{v(j)} FV''_{(j)} \sin\theta + \sum_{k=1}^{N_w} q_{w(k)} FW''_{(k)} \cos\theta - \sum_{k=1}^{N_w}\sum_{\ell=1}^{N_\phi} q_{w(k)} q_{\phi(\ell)} FW''_{(k)} F\phi_{(\ell)} \sin\theta - \sum_{j=1}^{N_v}\sum_{\ell=1}^{N_\phi} q_{v(j)} q_{\phi(\ell)} FV''_{(j)} F\phi_{(\ell)} \cos\theta - \sum_{j=1}^{N_v}\sum_{j1=1}^{N_v}\sum_{k=1}^{N_w} q_{v(j)} q_{v(j1)} q_{w(k)} FV''_{(j)} FV'_{(j1)} FW'_{(k)} \cos\theta \quad (27b)$$

$$\sum_{\ell=1}^{N_{\phi e}} q_{\phi e(\ell)} F\phi'_{e(\ell)} = \sum_{\ell=1}^{N_\phi} q_{\phi(\ell)} F\phi'_{(\ell)}$$
$$+ \sum_{j=1}^{N_v} \sum_{k=1}^{N_w} q_{v(j)} q_{w(k)} FV''_{(j)} FW'_{(k)}$$
$$- \sum_{j=1}^{N_v} \sum_{j1=1}^{N_v} \sum_{\ell=1}^{N_\phi} q_{v(j)} q_{v(j1)} q_{\phi(\ell)} FV''_{(j)} FV'_{(j1)} F\phi_{(\ell)}$$
$$+ \sum_{k=1}^{N_w} \sum_{k1=1}^{N_w} \sum_{\ell=1}^{N_\phi} q_{w(k)} q_{w(k1)} q_{\phi(\ell)} FW''_{(k)} FW'_{(k1)} F\phi_{(\ell)}$$
$$- \sum_{j=1}^{N_v} \sum_{k=1}^{N_w} \sum_{\ell=1}^{N_\phi} \sum_{\ell1=1}^{N_\phi} q_{v(j)} q_{w(k)} q_{\phi(\ell)} q_{\phi(\ell1)}$$
$$\times FV'_{(j)} FW''_{(k)} F\phi_{(\ell)} F\phi_{(\ell1)} \tag{27c}$$

The different methods of weighted residuals[13] seem to be natural candidates for solving Eqs. (27). Thus, for example, the collocation method can be applied quite easily. In the present paper a Galerkin method will be used.

According to this method, Eqs. (27a–c) are multiplied by the components of the principal curvature $FV''_{e(j)}$, $FW''_{e(k)}$, and $F\phi'_{e(\ell)}$, respectively. Then both sides of the equations are integrated with respect to x, from $x = 0$ to L. This results in a nonlinear algebraic transformation between the vectors of coefficients of the two systems of displacement components. This transformation is described by the following matrix notation:

$$[D1]\{q_e\} = [[D2] + [D3]]\{q\} \tag{28}$$

Here $\{q\}$ is the vector of coefficients of the Cartesian displacement components, defined as follows:

$$\{q\}^t = \langle q_{v(1)}, \ldots, q_{v(j)}, \ldots, q_{v(Nv)}, q_{w(1)}, \ldots, q_{w(k)}, \ldots,$$
$$q_{w(N)}, q_{\phi(1)}, \ldots, q_{\phi(\ell)}, \ldots, q_{\phi(N\phi)} \rangle \tag{29}$$

The dimension of this vector is N, where

$$N = N_v + N_w + N_\phi \tag{30}$$

Here, $[D1]$ is a square matrix of order N_e, and $[D2]$ and $[D3]$ are rectangular matrices of order $N_e \times N$. These three matrices contain all the Galerkin integrals resulting from the solution of Eqs. (27).

The matrices $[D1]$ and $[D2]$ represent the linear contributions; i.e., they are not functions of the vector $\{q\}$. The matrix $[D3]$ contains the nonlinear effects [all of the nonlinear terms on the right-hand side of Eqs. (27)] and its nonzero terms are functions of the elements of the vector $\{q\}$.

Matrices $[D1]$, $[D2]$, and $[D3]$ all result from Galerkin integrals in this procedure, and expressions for them are given in Ref. 17.

Substitution of Eq. (28) into Eq. (20) implies that

$$U_E = \tfrac{1}{2}\{q\}^t[K]\{q\} \tag{31}$$

Where the generalized stiffness matrix $[K]$ is given by

$$[K] = [K_1] + [K_2] \tag{32a}$$

$$[K_1] = [D2]^t[D4][D2] \tag{32b}$$

$$[K_2] = [D3]^t[D4][D2] + [D2]^t[D4][D3]$$
$$+ [D3]^t[D4][D3] \tag{32c}$$

$$[D4] = [D1]^{-1^t}[K_e][D1]^{-1} \tag{32d}$$

The matrix $[K_1]$ contains the linear parts of $[K]$, while $[K_2]$ incorporates the nonlinear contributions.

V. Potential of the Applied Loads

The rod is considered to be acted upon by a conservative distributed force $\bar{p}$ per unit length. This force is described by its components as follows:

$$\bar{p} = p_x\hat{e}_x + p_y\hat{e}_y + p_z\hat{e}_z \tag{33}$$

If it is assumed that this force acts at the elastic axis, then the potential of the applied force U_p becomes (the small warping displacements are neglected here)

$$U_p = -\int_0^L (p_x u + p_y v + p_z w)\,\mathrm{d}x \tag{34}$$

As one should recall, the assumption of inextensional bending has been adopted previously. Therefore, u is not an independent variable. Instead $u_{,x}$ is obtained by using Eqs. (3a), (4), (24a), and (24b) after $\bar{\epsilon}_c$ is made equal to zero. Thus,

$$u_{,x} = -\tfrac{1}{2}(v^2_{,x} + w^2_{,x}) + \underline{y_c(v_{,xx} + w_{,xx}\phi)}$$
$$+ \underline{Z_c(w_{,xx} - v_{,xx}\phi)} \tag{35}$$

In most practical cases, y_c and z_c are small enough that the underlined terms can be neglected. If this is not the case then these terms can be retained, increasing the number of calculations to be performed.

Since v, w, and ϕ are expressed in series form [Eqs. (26)], it will also be convenient to describe u in the same manner (a similar approach has been used in Refs. 6 and 14):

$$u = \sum_{n=1}^{N_u} q_{u(n)} FU_{(n)} \tag{36}$$

The functions $FU_{(n)}$ ensure that the boundary conditions on u are satisfied, and N_u is the number of elements in the series. In order to find the coefficients $q_{u(n)}$, Eqs. (26a), (26b), and (36) are substituted into Eq. (35). Then, according to the Galerkin method, the equation is multiplied on both sides by $FU'_{(n)}$ and integration is carried out between $x = 0$ and L. These steps result in a system of N_u equations in the N_u unknowns, $q_{u(n)}$:

$$\sum_{n=1}^{N_u} G14_{(nm,n)} q_{u(n)} = \sum_{j1=1}^{N_v} \sum_{j=1}^{N_v} G15_{(nm,j,j1)} q_{v(j)} q_{v(j1)}$$
$$+ \sum_{k1=1}^{N_w} \sum_{k=1}^{N_w} G16_{(nm,k,k1)} q_{w(k)} q_{w(k1)}, \quad 1 \le nm \le N_u \tag{37}$$

The integrals $G14$–$G16$ are defined by

$$G14_{(nm,n)} = \int_0^L FU'_{(nm)} FU'_{(n)}\,\mathrm{d}x \tag{38a}$$

$$G15_{(nm,j,j1)} = -\frac{1}{2}\int_0^L FU'_{(nm)} FV'_{(j)} FV'_{(j1)}\,\mathrm{d}x \tag{38b}$$

$$G16_{(nm,k,k1)} = -\frac{1}{2}\int_0^L FU'_{(nm)} FW'_{(k)} FW'_{(k1)}\,\mathrm{d}x \tag{38c}$$

The N_u equations (37) are solved for $q_{u(n)}$, yielding the following expressions:

$$q_{u(n)} = \sum_{j1=1}^{N_v} \sum_{j=1}^{N_v} H1_{(n,j,j1)} q_{v(j)} q_{v(j1)} + \sum_{k1=1}^{N_w} \sum_{k=1}^{N_w} H2_{(n,k,k1)} q_{w(k)} q_{w(k1)} \tag{39}$$

where

$$H1_{(n,j,j1)} = \sum_{nm=1}^{N_u} R_{(n,nm)} G15_{(nm,j,j1)} \tag{40a}$$

$$H2_{(n,k,k1)} = \sum_{nm=1}^{N_u} R_{(n,nm)} G16_{(nm,k,k1)} \tag{40b}$$

The terms $R_{(n,nm)}$ are the elements of the matrix $[R]$, which is defined as

$$[R] = [G14]^{-1} \tag{41}$$

The matrix $[G14]$ is the matrix of coefficients formed by the left-hand sides of the N_u equations (37).

Substitution of Eqs. (26a), (26b), and (36) into Eq. (31), yields the following expression for U_p:

$$U_p = -\sum_{n=1}^{N_u} q_{u(n)} PU_{(n)} - \sum_{j=1}^{N_v} q_{v(j)} PV_{(j)} - \sum_{k=1}^{N_w} q_{w(k)} PW_{(k)} \tag{42}$$

where

$$PU_{(n)} = \int_0^L p_x FU_{(n)} \, \mathrm{d}x \tag{43a}$$

$$PV_{(j)} = \int_0^L p_y FV_{(j)} \, \mathrm{d}x \tag{43b}$$

$$PW_{(k)} = \int_0^L p_z FW_{(k)} \, \mathrm{d}x \tag{43c}$$

Substitution of Eq. (39) into Eq. (42) and carrying out the summation with respect to n yield the final expression for the potential of the applied loads:

$$U_p = -\sum_{j=1}^{N_v} q_{v(j)} \left[PV_{(j)} + \sum_{j1=1}^{N_v} q_{v(j1)} PUV_{(j,j1)} \right] - \sum_{k=1}^{N_w} q_{w(k)} \left[PW_{(k)} + \sum_{k1=1}^{N_w} q_{w(k1)} PUW_{(k,k1)} \right] \tag{44}$$

where

$$PUV_{(j,j1)} = \sum_{n=1}^{N_u} H1_{(n,j,j1)} PU_{(n)} \tag{45a}$$

$$PUW_{(k,k1)} = \sum_{n=1}^{N_u} H2_{(n,k,k1)} PU_{(n)} \tag{45b}$$

VI. Equilibrium Equations

The total potential of the deformed rod V is given by

$$V = U_E + U_p \tag{46}$$

If the rod is in equilibrium then V provides a stationary value. This implies that

$$\frac{\partial V}{\partial q_{v(j)}} = \frac{\partial V}{\partial q_{w(k)}} = \frac{\partial V}{\partial q_{\phi(\ell)}} = 0 \tag{47}$$

Equation (47) yields a system of N-coupled equilibrium equations. The differentiation of V in the nonlinear case is more complicated than in the linear case[4] and deserves special care, as will be shown subsequently.

Substitution of Eqs. (31) and (44) into Eq. (46) and using Eq. (32a) yield the following system of equilibrium equations:

$$[[K_1] + [K_2] + [K_3]]\{q\} = \{f_p\} - \{f_s\} \tag{48}$$

The matrices $[K_1]$ and $[K_2]$ are of order $N \times N$ and symmetric matrices defined in Sect. IV. It should be recalled that $[K_2]$ is a function of $\{q\}$. The matrix $[K_3]$ is also an $N \times N$ matrix, which is obtained as a result of differentiation of Eq. (44). This matrix is composed of two submatrices as follows:

$$[K_3] = \begin{bmatrix} [D8] & & 0 \\ & [D9] & \\ 0 & & 0 \end{bmatrix} \tag{49}$$

Matrix $[D8]$ is of order $N_v \times N_v$, the elements of which are the terms $(-2PUV_{(j,j1)})$. Matrix $[D9]$ is of order $N_w \times N_w$, the elements of which are the terms $(-2PUW_{(k,k1)})$.

The vector $\{f_p\}$ represents an N-dimensional load, defined as

$$\{f_p\}^T = \langle PV_{(1)}, \ldots, PV_{(j)}, \ldots, PV_{(N_v)}, PW_{(1)}, \ldots, PW_{(k)}, \ldots, PW_{(N_w)}, 0..0 \rangle \tag{50}$$

The vector $\{f_s\}$ is also N-dimensional and is a result of the differentiation of $[K_2]$ with respect to $q_{v(j)}$, $q_{w(k)}$, or $q_{\phi(\ell)}$. Therefore, this vector represents, in fact, a nonlinear structural contribution which, when transferred to the right-hand side of the equations, can be looked upon as an additional quasiload. If the terms of the vector $\{f_s\}$ are denoted $f_{s(r)}$, then they are defined as

$$f_{s(r)} = \tfrac{1}{2}\{q\}^t [Q_{(r)}]\{q\}, \qquad 1 \le r \le N \tag{51}$$

where

$$[Q_{(r)}] = \left[\frac{\partial}{\partial q_{(r)}} [K_2] \right] = [DD3_{(r)}]^t [D4][D2] + [D2]^t [D4][DD3_{(r)}] + [DD3_{(r)}]^t [D4][D3] + [D3]^t [D4][DD3_{(r)}] \tag{52a}$$

and

$$[DD3_{(r)}] = \left[\frac{\partial}{\partial q_{(r)}} [D3] \right] \tag{52b}$$

The matrices $[DD3_{(r)}]$ are calculated by direct analytic differentiation of $[D3]$. Exact expressions for all of these matrices are given in Appendix B of Ref. 17.

VII. Solution Procedure

As a first step, the structural properties of the rod and the "principal curvature" series are defined. If the natural, uncoupled, free-vibration frequencies of the analog rod are not available, the following input is required:

1) Distributions of $(EI_{\xi\xi})$, $(EI_{\eta\eta})$, and (GJ) along the rod.

2) N_{ve}, N_{we}, and $N_{\phi e}$ shape functions $FV''_{e(j)}$, $FW''_{e(k)}$, and $F\phi'_{e(\ell)}$, respectively.

3) Angle θ as a function of the spanwise coordinate.

With the preceding input, the coefficients $B1$, $B2$, and $B3$ can be calculated according to Eqs. (16), and the generalized stiffness matrix assembled according to Eq. (22).

If the natural uncoupled modes of free vibration of the analog rod are used as generalized coordinates, then the following input is required:

1) N_{ve} chordwise modes and frequencies $[FV_{e(j)}, \omega_{ve(j)}]$.

2) N_{we} beamwise modes and frequencies $[FW_{e(k)}, \omega_{we(k)}]$.

3) $N_{\phi e}$ torsional modes and frequencies $[F\phi_{e(\ell)}, \omega_{\phi e(\ell)}]$.

4) Mass distributions (m and MI_p) and angle θ as functions of the spanwise coordinate.

With this data the matrix $[K_e]$ is calculated, using Eq. (23).

At this stage, N_v $FV_{(j)}$, N_w $FW_{(k)}$, N_ϕ $F\phi_{(\ell)}$, and N_u $FU_{(h)}$ functions are determined. For most of the combinations of boundary conditions usually encountered, it is very convenient to choose $FV_{(j)}$, $FW_{(k)}$, and $F\phi_{(\ell)}$, identical to $FV_{e(j)}$, $FW_{e(k)}$, and $F\phi_{e(\ell)}$, respectively, as noted previously.

The solution begins with calculation of the Galerkin integrals. If one plans repeated calculations, then storing these integrals on a disk or tape and reusing them will save computing time as compared to calculating the integrals each time. These integrations can be performed using any convenient integrating scheme.

At this stage, the matrices $[D1]$, $[D2]$, $[D4]$, and $[K1]$ are assembled. Again, computing time will be conserved by keeping these matrices on a disk or tape when repeated runs for a specific rod are planned.

For a certain load distribution, the matrix $[K_3]$ and vector $\{f_p\}$ are calculated using Eqs. (43), (49), and (50). The solution for this load is obtained as follows:

1) An initial value of $\{q\}$ is assumed. This can be, for example, the solution for a lower load or it can be chosen equal to zero.

2) Calculate the matrix $[D3]$.

3) Calculate the matrix $[K_2]$ using Eq. (32c).

4) Calculate the vector $\{f_s\}$ using Eq. (57).

5) Solve the system of Eqs. (48).

6) Check for convergence (between the assumed and obtained value of $\{q\}$).

If convergence is not reached, assume another value of $\{q\}$ and start from step 2 again. The new assumed value of $\{q\}$ can be chosen as the last result for $\{q\}$ or a value between the two last results for $\{q\}$, using a relaxation factor.

VIII. Discussion and Conclusions Relative to the Theoretical Derivation

In the previous sections, the derivation of equations for analyzing the nonlinear coupled bending-torsion of pretwisted rods and methods of solution have been described. The method combines principal curvature transformation and generalized coordinates techniques. It seems that this combination yields a very efficient method, as is emphasized in what follows.

1) The structural properties of the rod ($EI_{\eta\eta}$, $EI_{\zeta\zeta}$, GJ) influence only the matrix $[K_e]$. This matrix can be calculated by using either Eq. (22) or (23). It seems that, if possible, it is very advantageous to use the natural modes of vibration of the "analog rod" as generalized coordinates and Eq. (23) for assembling the matrix $[K_e]$. For cases where repeated changes of the distributions of the structural properties of the rod are required, however, the properties of the analog rod will be changed, thus changing the natural modes of vibration of that rod. Such changes, in cases where these modes are used as the generalized coordinates, require a complete new calculation of all the Galerkin integrals. Therefore, in such cases, it seems that it is preferable to choose one set of FV''_e, FW''_e, $F\phi'_e$, FV, FW, and $F\phi$ functions (satisfying the appropriate boundary conditions) for use throughout the study. Thus, one would have to recalculate the integral $B1$, $B2$, or $B3$ [see Eqs. (16)] as a result of changes in the structural properties and use the new values to reassemble $[K_e]$ using Eq. (22). On the other hand, however, all of the Galerkin integrals will not be changed, thereby saving a relatively large amount of computation. It is clear that, for changes where the frequencies are altered but not the mode shapes (as, for example, varying properties but not their relative spanwise distribution), it may still be advantageous to use these modes as generalized coordinates.

2) Changing pretwist distribution (angle θ) requires recalculation of part of the Galerkin integrals. Since many of the integrals are not functions of θ (those which are associated with elastic twist), it seems advisable to group integrals according to whether they are functions of θ or not. Thus the program can more easily be set up to recalculate, when changes in θ are introduced, only those integrals influenced by θ change.

3) The only elements of the method that are changed as a result of using another nonlinear model are the matrix $[D3]$ and the matrices $[DD3_{(r)}]$. Therefore, if one is interested in using a nonlinear model different from the one presented here (probably dealing with higher-order nonlinearities), only $[D3]$ and $[DD3_{(r)}]$ have to be changed, while the rest of the computer code remains the same. To illustrate the relative ease with which this can be done, a different nonlinear model is derived in the Appendix. While the derivations of the previous sections are based on the assumption of small strains and moderate elastic rotations (meaning that strains and products of elastic rotations are neglected compared to unity), the Appendix pertains to cases in which elastic rotations due to bending are large enough that their products cannot be neglected compared with unity. The derivation of the equations and their implementation in the existing computer code took only 1 man-day. Trying to do the same task using other methods of analysis (see, for example, Ref. 1, 5, or 7) would have required efforts orders of magnitude greater. The effects of using the more accurate model of the Appendix are presented in the second part of the paper.[15] It should also be pointed out that, to include still higher-order nonlinearities, Eq. (35) must be replaced by a more accurate expression, in accordance with Eq. (A10). This will result in additional changes in the matrix $[K_3]$. [See Eq. (49).]

4) Another significant advantage of the new model is the fact that the derivations and the associated computer code are relatively simple. They do not require the kind of tedious derivation that usually results in very complicated, long equations with the attendant increased risk that errors will be introduced into the model. Perhaps the best indication that the present method is simpler (though by no means less accurate) than others is the fact that it has not been necessary to invoke a variety of different assumptions widely used previously to simplify matters. Examples of assumptions used earlier, but not here, include: the stiffness in bending in one direction is much larger than the other[1,5] and, therefore, one component of the transverse displacement is usually larger than the other; uniform distribution of the properties along the blade and no pretwist.[2] Any effort to avoid such assumptions in the earlier methods results in an inordinate increase in the derivation effort and the length of the final results.

5) One of the main advantages of using generalized coordinates is the ability to get an accurate solution with a relatively small number of degrees of freedom. In such cases, the generalized coordinates are functions that are defined along the whole length of the rod. On the other hand, it is also possible to use as the generalized coordinates functions, which are defined only along small segments of the rod (spline and other kinds of functions). If the generalized coordinates in the present case (describing FV''_e, FW''_e, $F\phi'_e$, FV, FW, $F\phi$, FU) are chosen as such "local functions," then a "finite element" model can be obtained with all of the benefits of such a model.

6) As indicated previously, it is very easy to extend the present model to include different types of geometrical nonlinearities. It would seem, then, that it would be possible to extend the model using a similar approach and, thereby, make it capable of representing the behavior of curved rods.

From all of the preceding, the present method appears to have many advantages and significant potential for extension. The proof of a method's utility, of course, is in its accuracy and efficiency in solving various problems; in this case, involving nonlinear behavior of rods. In the second part of the paper,[15] results obtained using the new method to solve several nonlinear problems of rods are presented and discussed.

Appendix A

The derivation in the preceding text deals with the case of small strains and moderate elastic rotations. This means that strains and products of the elastic rotations are neglected compared with unity. While many practical cases fall within the limits of these assumptions, there are also cases where the elastic rotations are large enough that their products are no longer negligible compared with unity. The purpose of this Appendix is to extend the present model to include a certain subclass of such cases.

The text defines $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ as a triad of orthogonal unit vectors in the directions of the coordinate lines x, y, and z, respectively. As a result of the deformation, this triad, at every point along the blade axis, is transformed into a new orthogonal triad $\hat{e}_{x1}$, $\hat{e}_{y1}$, and $\hat{e}_{z1}$. This transformation is described by Euler angles, where the following sequence is chosen: rotation θ_z about $\hat{e}_z$, followed by a rotation θ_y about $\hat{e}_y$ (at its new direction), followed by a rotation θ_x about $\hat{e}_x$ (at its new direction). In this case,[6,7,14]

$$\begin{Bmatrix} \hat{e}_{x1} \\ \hat{e}_{y1} \\ \hat{e}_{z1} \end{Bmatrix} = [T_{E1E}] \begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{Bmatrix} \tag{A1}$$

where

$$[T_{E1E}] \triangleq \begin{bmatrix} \cos\theta_y \cos\theta_z & \cos\theta_y \sin\theta_z & -\sin\theta_y \\ (\sin\theta_x \sin\theta_y \cos\theta_z - \cos\theta_x \sin\theta_z) & (\cos\theta_x \cos\theta_z + \sin\theta_x \sin\theta_y \sin\theta_z) & \sin\theta_x \cos\theta_y \\ (\cos\theta_x \sin\theta_y \cos\theta_z + \sin\theta_x \sin\theta_z) & (-\sin\theta_x \cos\theta_z + \cos\theta_x \sin\theta_y \sin\theta_z) & \cos\theta_x \cos\theta_y \end{bmatrix} \tag{A2}$$

It can be shown (see Refs. 6, 7, and 14, taking the curvature χ_y equal to zero) that the Euler angles can be expressed as functions of the displacement components u, v, and w and the cross-sectional rotation ϕ about the elastic axis as follows:

$$\theta_x \equiv \phi \tag{A3}$$

$$\sin\theta_y = -w_{,x}/D; \; \cos\theta_y = D_1/D \tag{A4}$$

$$\sin\theta_z = v_{,x}/D_1; \; \cos\theta_z = (1+u_{,x})/D_1 \tag{A5}$$

where

$$D_1 = \sqrt{D^2 - w_{,x}^2} \tag{A6}$$

Equations (A3–A6) are exact expressions. D is the ratio between the length of an element of the elastic axis after and before the deformation. For normal structural materials, strains can be taken as small and, in fact, the assumption of inextensibility adopted. The following equation can then be used:

$$D \simeq 1 \tag{A7}$$

To this small strain assumption another is added; namely, that the angle ϕ is small enough that

$$\sin\phi \simeq \phi; \; \cos\phi \simeq 1 \tag{A8}$$

in using Eqs. (A3–A8), the transformation matrix $[T_{E1E}]$ then becomes

$$[T_{E1E}] = \begin{bmatrix} (1+u_{,x}) & v_{,x} & w_{,x} \\ -\dfrac{v_{,x} + \phi w_{,x}(1+u_{,x})}{\sqrt{1-w_{,x}^2}} & \dfrac{1+u_{,x} - \phi v_{,x} w_{,x}}{\sqrt{1-w_{,x}^2}} & \phi\sqrt{1-w_{,x}^2} \\ \dfrac{-w_{,x}(1+u_{,x}) + \phi v_{,x}}{\sqrt{1-w_{,x}^2}} & \dfrac{-\phi(1+u_{,x}) + v_{,x} w_{,x}}{\sqrt{1-w_{,x}^2}} & \sqrt{1-w_{,x}^2} \end{bmatrix} \tag{A9}$$

Equation (A7) implies (for details, see Refs. 6, 7, and 14) that

$$u_{,x} = \sqrt{1 - v_{,x}^2 - w_{,x}^2} - 1 \tag{A10}$$

At this stage, a third assumption is adopted; namely, that terms higher than third order in the displacements or their derivatives are small and can be ignored. Thus, if the square root in Eq. (A10) is expanded using Taylor series, and the third assumption is used, the following expression is obtained:

$$u_{,x} = -\tfrac{1}{2}\left(v_{,x}^2 + w_{,x}^2\right) \tag{A11}$$

Differentiating Eq. (A11) implies that

$$u_{,xx} = -w_{,xx}w_{,x} - v_{,xx}v_{,x} \tag{A12}$$

Using the definition of the curvature component K_y [see Eq. (5a)] and Eqs. (A1), (A9), and (A12) the following expression is obtained:

$$K_y = \hat{e}_{y1} \cdot \hat{e}_{x1,x} = \frac{1}{\sqrt{1-w_{,x}^2}} \times \{ (w_{,xx}w_{,x} + v_{,xx}v_{,x})[\phi w_{,x}(1+u_{,x}) + v_{,x}] + v_{,xx}(1 + u_{,x} - \phi v_{,x}w_{,x}) + w_{,xx}\phi(1 - w_{,x}^2) \}$$

Neglecting terms of orders higher than the third yields the final expression for K_y,

$$K_y = v_{,xx} + w_{,xx}\phi + \tfrac{1}{2}v_{,xx}v_{,x}^2 + v_{,x}w_{,xx}w_{,x} \tag{A13}$$

Similarly, Eqs. (5b), (A1), (A9), and (A12) lead to the following equation for the curvature component K_z:

$$K_z = \hat{e}_{z1} \cdot \hat{e}_{x1,x} = \frac{1}{\sqrt{1-w_{,x}^2}} \times \{ (w_{,xx}w_{,x} + v_{,xx}v_{,x})[w_{,x}(1+u_{,x}) - \phi v_{,x}] - v_{,xx}[\phi(1+u_{,x}) + v_{,x}w_{,x}] + w_{,xx}(1 - w_{,x}^2) \}$$

Retaining only terms up to third order yields

$$K_z = w_{,xx} - v_{,xx}\phi + \tfrac{1}{2} w_{,xx} w_{,x}^2 \qquad (A14)$$

Finally, the expression for the twist, after neglecting the small terms, becomes

$$T = \hat{e}_{y1,x} \cdot \hat{e}_{z1} = \phi_{,x} + v_{,xx} w_{,x} \qquad (A15)$$

It is interesting to note that the third-order terms that appear and are retained during the derivation of T ultimately cancel one another so that the final expression includes only first- and second-order terms.

Substituting Eqs. (A13–A15) into Eqs. (12) yields the following transformations between the principal curvatures and the Cartesian displacement components:

$$v_{e,xx} = \left(v_{,xx} + w_{,xx}\phi + \tfrac{1}{2} v_{,xx} v_{,x}^2 + v_{,x} w_{,xx} w_{,x} \right) \cos\theta + \left(w_{,xx} - v_{,xx}\phi + \tfrac{1}{2} w_{,xx} w_{,x}^2 \right) \sin\theta \qquad (A16a)$$

$$w_{e,xx} = -\left(v_{,xx} + w_{,xx}\phi + \tfrac{1}{2} v_{,xx} v_{,x}^2 + v_{,x} w_{,xx} w_{,x} \right) \sin\theta + \left(w_{,xx} - v_{,xx}\phi + \tfrac{1}{2} w_{,xx} w_{,x}^2 \right) \cos\theta \qquad (A16b)$$

$$\phi_{e,x} = \phi_{,x} + v_{,xx} w_{,x} \qquad (A16c)$$

Following steps identical to those described in the text, the same expression for the strain energy and potential of applied loads is obtained. The only changes due to keeping products of rotations up to third order appear in the matrix $[D3]$. All of the details concerning the matrices $[D3]$ and $[DD3_{(r)}]$ are given in Appendix C of Ref. 17.

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